A Spacetime Foam approach to the cosmological constant and entropy

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Abstract

A simple model of spacetime foam, made by N wormholes in a semiclassical approximation, is taken under examination. The Casimir-like energy of the quantum fluctuation of such a model and its probability of being realized are computed. Implications on the Bekenstein-Hawking entropy and the cosmological constant are considered.

I. INTRODUCTION

The problem of merging General Relativity with Quantum Field Theory is known as Quantum Gravity. Many efforts to give meaning to a quantum theory of gravity have been done. Unfortunately until now, such a theory does not exist, even if other approaches like string theory and the still unknown M-theory seem to receive a wide agreement in this direction. Nevertheless a direct investigation of Quantum Gravity shows a lot of interesting

aspects. One of these is that at the Planck scale, spacetime could be subjected to topology and metric fluctuations [1]. Such a fluctuating spacetime is known under the name of "*spacetime foam*" which can be taken as a model for the quantum gravitational vacuum[1]. At this scale of lengths (or energies) quantum processes like black hole pair creation could become relevant. To establish if a foamy spacetime could be considered as a candidate for a Quantum Gravitational vacuum, we can examine the structure of the effective potential for such a spacetime. It has been shown that flat space is the classical minimum of the energy for General Relativity [3]. However there are indications that flat space is not the true ground state when a temperature is introduced, at least for the Schwarzschild space in absence of matter fields [4]. It is also argued that when gravity is coupled to N conformally invariant scalar fields the evidence that the ground-state expectation value of the metric is flat space is false [5]. Moreover it is also believed that in a foamy spacetime, general relativity can be renormalized when a density of virtual black holes is taken under consideration coupled to N fermion fields in a $1/N$ expansion [6]. With these examples at hand, we have investigated the possibility of having a ground state different from flat space even at zero temperature and what we have discovered is that there exists an imaginary contribution to the effective potential (more precisely effective energy) at one loop in a Schwarzschild background, that it means that flat space is unstable [7]. What is the physical interpretation associated to this instability. We can begin by observing that the "simplest" quantum process approximating a foamy spacetime, in absence of matter fields, could be a black hole pair creation of the neutral type. One possibility of describing such a process is represented by the Schwarzschild-de Sitter metric (SdS) which asymptotically approaches the de Sitter metric. Its degenerate or extreme version is best known as the Nariai metric [8]. Here we have an external background, the cosmological constant Λ, which gives a nonzero probability of having a neutral black

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[1]It is interesting to note that there are also indications on how a foamy spacetime can be tested experimentally [2].

hole pair produced with its components accelerating away from each other. Nevertheless this process is believed to be highly suppressed, at least for $\Lambda \gg 1$ in Planck's units [9]. In any case, metrics with a cosmological constant have different boundary conditions compared to flat space. The Schwarzschild metric is the only case available. Here the whole spacetime can be regarded as a black hole-anti-black hole pair formed up by a black hole with positive mass M in the coordinate system of the observer and an *anti black-hole* with negative mass $-M$ in the system where the observer is not present. In this way we have an energy preserving mechanism, because flat space has *zero energy* and the pair has zero energy too. However, in this case we have not a cosmological *force* producing the pair: we have only pure gravitational fluctuations. The black hole-anti-black hole pair has also a relevant pictorial interpretation: the black hole with positive mass M and the *anti black-hole* with negative mass $-M$ can be considered the components of a virtual dipole with zero total energy created by a large quantum gravitational fluctuation [10]. Note that this is the only physical process compatible with the energy conservation. The importance of having the same energy behaviour (*asymptotic*) is related to the possibility of having a spontaneous transition from one spacetime to another one with the same boundary condition [11]. This transition is a decay from the false vacuum to the true one [12,13]. However, if we take account of a pair of neutral black holes living in different universes, there is no decay and more important no temperature is involved to change from flat to curved space. To see if this process is realizable we need to compute quantum corrections to the energy stored in the boundaries. These quantum corrections are pure gravitational vacuum excitations which can be measured by the Casimir energy, formally defined as

$$E_{Casimir}\left[\partial\mathcal{M}\right] = E_0\left[\partial\mathcal{M}\right] - E_0\left[0\right], \tag{1}$$

where E_0 is the zero-point energy and $\partial\mathcal{M}$ is a boundary. For zero temperature, the idea underlying the Casimir effect is to compare vacuum energies in two physical distinct configurations. For gravitons embedded in flat space, the one-loop contribution to the zero point energy (ZPE) is

$$2 \cdot \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \sqrt{k^2}. \tag{2}$$

This term is UV divergent, and its effect is equivalent to inducing a divergent "*cosmological constant*" [14,15]

$$\Lambda_{ZPE} = 8\pi G \rho_{ZPE} = 4\pi G \int \frac{d^3k}{(2\pi)^3} \sqrt{k^2}. \tag{3}$$

However, it is likely that a natural cutoff of the order of the inverse Planck length comes into play. If this is the case we expect

$$\Lambda_{ZPE} = c \frac{8\pi}{l_P^2}, \tag{4}$$

where c is a dimensionless constant. There are some observational data coming from the Friedmann-Robertson-Walker cosmology constraining the cosmological constant [16]. An estimate of these ones is

$$\begin{array}{c} \Lambda \lesssim 10^{-122} l_P^{-2} \\ c \lesssim 10^{-123} \end{array} . \tag{5}$$

What is the relation between Λ_{ZPE} and the neutral black hole pair production? Since the pair produced is mediated by a three-dimensional wormhole with its own ZPE showing a Casimir-like energy and an imaginary part, if we enlarge the number of wormholes (or equivalently the pair creation number) to a certain value N. In section II, we will prove that the imaginary part disappears and the system from unstable turns to be stable. In Ref. [17] a spacetime foam model made by N *coherent* wormholes has been proposed. In that paper, we have computed the energy density of gravitational fluctuations reproducing the same behavior conjectured by Wheeler during the sixties on dimensional grounds [1]. As an application of the model proposed in Ref. [17], in Ref. [18] we have computed the spectrum of a generic area and as a particular case we have considered the de Sitter geometry. The result is a *quantization* process whose quanta can be identified with wormholes of Planckian size. In this paper, we would like to continue the investigation of such a model of *spacetime foam*, by looking at the problem of the cosmological constant, from the Casimir-like energy

point of view, and as a consequence of the *quantization* process of Ref. [18], we will give indications about the black hole mass quantization. The rest of the paper is structured as follows, in section II and III we briefly recall the results reported in Refs. [17,18], by looking at the mass quantization process. In section IV we approach the cosmological constant problem. We summarize and conclude in section V. Units in which $\hbar = c = k = 1$ are used throughout the paper.

II. SPACETIME FOAM: THE MODEL

In the one-wormhole approximation we have used an eternal black hole, to describe a complete manifold $\mathcal{M}$, composed of two wedges $\mathcal{M}_+$ and $\mathcal{M}_-$ located in the right and left sectors of a Kruskal diagram. The spatial slices Σ represent Einstein-Rosen bridges with wormhole topology $S^2 \times R^1$. Also the hypersurface Σ is divided in two parts Σ_+ and Σ_- by a bifurcation two-surface S_0. The line element we shall consider is

$$ds^2 = -N^2(r)\, dt^2 + \frac{dr^2}{1 - \frac{2MG}{r}} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{6}$$

or by defining the proper distance from the throat $dy = \pm dr/\sqrt{1 - \frac{2MG}{r}}$, metric (6) becomes

$$ds^2 = -N^2 dt^2 + g_{yy} dy^2 + r^2(y)\, d\Omega^2. \tag{7}$$

N, g_{yy}, and r are functions of the radial coordinate y continuously defined on $\mathcal{M}$. The throat of the bridge is at $r = 2MG$ or $y = 0$. We choose y to be positive in Σ_+ and negative in Σ_-. If we make the identification $N^2 = 1 - 2MG/r$, the line element (6) reduces to the Schwarzschild metric written in another form. The boundaries ${}^2S_+$ and ${}^2S_-$ are located at coordinate values $y = y_+$ and $y = y_-$ respectively. The physical Hamiltonian defined on Σ is

$$H_P = H - H_0 = \frac{1}{16\pi l_p^2}\int_\Sigma d^3x\left(N\mathcal{H} + N_i\mathcal{H}^i\right) + H_{S_+} - H_{S_-} \tag{8}$$

where $l_p^2 = G$ and $H_{S_+} - H_{S_-}$ represents the boundary hamiltonian defined by

$$+ \frac{1}{8\pi l_p^2}\int_{S_+} d^2x N\sqrt{\sigma}\left(k-k^0\right) - \frac{1}{8\pi l_p^2}\int_{S_-} d^2x N\sqrt{\sigma}\left(k-k^0\right), \tag{9}$$

where σ is the two-dimensional determinant coming from the induced metric σ_{ab} on the boundaries $S_\pm$. The volume term contains two constraints

$$\begin{cases} \mathcal{H} = G_{ijkl}\pi^{ij}\pi^{kl}\left(\frac{l_p^2}{\sqrt{g}}\right) - \left(\frac{\sqrt{g}}{l_p^2}\right)R^{(3)} = 0 \\ \mathcal{H}^i = -2\pi^{ij}_{|j} = 0 \end{cases}, \tag{10}$$

both satisfied by the Schwarzschild and Flat metric on-shell, respectively. The supermetric is $G_{ijkl} = \frac{1}{2}\left(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl}\right)$ and $R^{(3)}$ denotes the scalar curvature of the surface Σ. In particular on the boundary we shall use the quasilocal energy definition. Quasilocal energy E_{ql} is defined as the value of the Hamiltonian that generates unit time translations orthogonal to the two-dimensional boundaries, i.e. [19–22]

$$E_{ql} = E_+ - E_-,$$

$$E_+ = \frac{1}{8\pi l_p^2}\int_{S_+} d^2x\sqrt{\sigma}\left(k-k^0\right)$$

$$E_- = -\frac{1}{8\pi l_p^2}\int_{S_-} d^2x\sqrt{\sigma}\left(k-k^0\right). \tag{11}$$

where $|N| = 1$ at both S_+ and S_-. E_{ql} is the quasilocal energy of a spacelike hypersurface $\Sigma = \Sigma_+ \cup \Sigma_-$ bounded by two boundaries ${}^3S_+$ and ${}^3S_-$ located in the two disconnected regions M_+ and M_- respectively. We have included the subtraction terms k^0 for the energy. k^0 represents the trace of the extrinsic curvature corresponding to embedding in the two-dimensional boundaries ${}^2S_+$ and ${}^2S_-$ in three-dimensional Euclidean space. By using the expression of the trace

$$k = -\frac{1}{\sqrt{h}}\left(\sqrt{h}n^\mu\right)_{,\mu}, \tag{12}$$

with the normal to the boundaries defined continuously along Σ as $n^\mu = \left(h^{yy}\right)^{\frac{1}{2}}\delta^\mu_y$, the value of k depends on the function $r_{,y}$, where we have assumed that the function $r_{,y}$ is positive for S_+ and negative for S_-. We obtain at either boundary that

$$k = \frac{-2r_{,y}}{r}. \tag{13}$$

The trace associated with the subtraction term is taken to be $k^0 = -2/r$ for B_+ and $k^0 = 2/r$ for B_-. Thus the quasilocal energy with subtraction terms included is

$$E_{ql} = (E_+ - E_-) = l_p^{-2} \left[(r\,[1 - |r_{,y}\,|])_{y=y_+} - (r\,[1 - |r_{,y}\,|])_{y=y_-} \right]. \tag{14}$$

It is easy to see that E_+ and E_- tend individually to the $\mathcal{ADM}$ mass M when the boundaries ${}^3B_+$ and ${}^3B_-$ tend respectively to right and left spatial infinity. It should be noted that the total energy is zero for boundary conditions symmetric with respect to the bifurcation surface, i.e.,

$$E = E_+ - E_- = M + (-M) = 0, \tag{15}$$

where the asymptotic contribution has been considered. We can recognize that the expression which defines quasilocal energy is formally of the Casimir type. Indeed, the subtraction procedure present in Eq.(11) describes an energy difference between two distinct situations with the same boundary conditions. The same behaviour appears in the entropy calculation for the physical system under examination. Indeed [22]

$$S_{tot} = S_+ - S_- = \frac{A^+}{4l_p^2} - \frac{A^-}{4l_p^2} \simeq \frac{A_H}{4l_p^2} - \frac{A_H}{4l_p^2} = 0, \tag{16}$$

where A^+ and A^- have the same meaning as E_+ and E_-. Note that for both entropy and energy this result is obtained at the tree level. In particular, the quasilocal energy can be interpreted as the tree level approximation of the Casimir energy. We can also see Eqs.(15) and (16) from a different point of view. In fact these equations say that flat space can be thought of as a composition of two pieces: the former, with positive energy, in the region Σ_+ and the latter, with negative energy, in the region Σ_-, where the positive and negative concern the bifurcation surface (hole) which is formed due to a topology change of the manifold. The most appropriate mechanism to explain this splitting seems to be a neutral black hole pair creation. To this purpose we begin by considering perturbations at Σ of the type

$$g_{ij}^{S,F} = \bar{g}_{ij}^{S,F} + h_{ij}, \tag{17}$$

where $\bar{g}_{ij}^{S}$ is the spatial part of the Schwarzschild background and $\bar{g}_{ij}^{F}$ is the spatial part of the Flat background. In this framework we compute the effective energy defined by

$$E = \min_{\{\Psi\}} \frac{\langle \Psi \left| H \right| \Psi \rangle}{\langle \Psi | \Psi \rangle} \tag{18}$$

with the help of the following rules

$$\begin{gathered} \int \left[\mathcal{D}h_{ij}\right] h_{ij}\left(\overrightarrow{x}\right) \mid \Psi\left\{h_{ij}+\bar{g}_{ij}\right\} \mid^2 = 0, \\ \frac{\int [\mathcal{D}h_{ij}] \int d^3x h_{ij}\left(\overrightarrow{x}\right) h_{kl}\left(\overrightarrow{y}\right) |\Psi\{h_{ij}+\bar{g}_{ij}\}|^2}{\int [\mathcal{D}h_{ij}] |\Psi\{h_{ij}+\bar{g}_{ij}\}|^2} = K_{ijkl}\left(\overrightarrow{x}, \overrightarrow{y}\right). \end{gathered} \tag{19}$$

In particular

$$E\left(M\right) = \frac{\left\langle \Psi \left| H^{Schw.} \right| \Psi \right\rangle}{\langle \Psi | \Psi \rangle} + \frac{\left\langle \Psi \left| H^{Schw.}_{boundary} \right| \Psi \right\rangle}{\langle \Psi | \Psi \rangle} \tag{20}$$

and

$$E\left(0\right) = \frac{\left\langle \Psi \left| H^{Flat} \right| \Psi \right\rangle}{\langle \Psi | \Psi \rangle} + \frac{\left\langle \Psi \left| H^{Flat}_{boundary} \right| \Psi \right\rangle}{\langle \Psi | \Psi \rangle} \tag{21}$$

so that the physical Hamiltonian is given by the difference (Casimir energy) $\Delta E\left(M\right)$

$$= E\left(M\right) - E\left(0\right) = \frac{\left\langle \Psi \left| H^{Schw.} - H^{Flat} \right| \Psi \right\rangle}{\langle \Psi | \Psi \rangle} + \frac{\langle \Psi \left| H_{ql} \right| \Psi \rangle}{\langle \Psi | \Psi \rangle}. \tag{22}$$

The quantity $\Delta E\left(M\right)$ is computed by means of a variational approach, where the WKB functionals are substituted with trial wave functionals. By restricting our attention to the graviton sector of the Hamiltonian approximated to second order, hereafter referred as $H_{|2}$, we define

$$E_{|2} = \frac{\left\langle \Psi^{\perp} \left| H_{|2} \right| \Psi^{\perp} \right\rangle}{\langle \Psi^{\perp} | \Psi^{\perp} \rangle},$$

where

$$\Psi^{\perp} = \Psi\left[h_{ij}^{\perp}\right] = \mathcal{N} \exp\left\{-\frac{1}{4l_p^2}\left[\left\langle \left(g-\bar{g}\right) K^{-1} \left(g-\bar{g}\right)\right\rangle_{x,y}^{\perp}\right]\right\}.$$

After having functionally integrated $H_{|2}$, we get

$$H_{|2} = \frac{1}{4l_p^2} \int_\Sigma d^3x \sqrt{g} G^{ijkl} \left[K^{-1\perp}(x,x)_{ijkl} + (\triangle_2)^a_j K^\perp(x,x)_{iakl} \right] \tag{23}$$

The propagator $K^\perp(x,x)_{iakl}$ comes from a functional integration and it can be represented as

$$K^\perp(\overrightarrow{x}, \overrightarrow{y})_{iakl} := \sum_N \frac{h^\perp_{ia}(\overrightarrow{x}) h^\perp_{kl}(\overrightarrow{y})}{2\lambda_N(p)}, \tag{24}$$

where $h^\perp_{ia}(\overrightarrow{x})$ are the eigenfunctions of

$$(\triangle_2)^a_j := -\triangle \delta^a_j + 2R^a_j. \tag{25}$$

This is the Lichnerowicz operator projected on Σ acting on traceless transverse quantum fluctuations and $\lambda_N(p)$ are infinite variational parameters. $\triangle$ is the curved Laplacian (Laplace-Beltrami operator) on a Schwarzschild background and R^a_j is the mixed Ricci tensor whose components are:

$$R^a_j = diag\left\{ \frac{-2MG}{r^3}, \frac{MG}{r^3}, \frac{MG}{r^3} \right\}. \tag{26}$$

The minimization with respect to λ and the introduction of a high energy cutoff Λ give to the Eq. (22) the following form

$$\Delta E(M) \sim -\frac{V}{32\pi^2} \left(\frac{3MG}{r_0^3} \right)^2 \ln\left(\frac{r_0^3 \Lambda^2}{3MG} \right), \tag{27}$$

where V is the volume of the system and r_0 is related to the minimum radius compatible with the wormhole throat. We know that classically, the minimum radius is achieved when $r_0 = 2MG$. However, it is likely that quantum processes come into play at short distances, where the wormhole throat is defined, introducing a *quantum* radius $r_0 > 2MG$. Nevertheless, since we are interested to probe the energy contribution near the Planck scale we can fix the value of $r_0 = l_p$ from now on. We now compute the minimum of $\Delta E(M)$, after having rescaled the variable M to a scale variable $x = 3MG/(r_0^3 \Lambda^2) = 3M/(l_p \Lambda^2)$. Thus

$$\Delta E(M) \to \Delta E(x, \Lambda) = \frac{V}{32\pi^2} \Lambda^4 x^2 \ln x$$

We obtain two values for x: $x_1 = 0$, i.e. flat space and $x_2 = e^{-\frac{1}{2}}$. At the minimum

$$\Delta E\left(x_2\right) = -\frac{V}{64\pi^2}\frac{\Lambda^4}{e}. \tag{28}$$

Nevertheless, there exists another part of the spectrum which has to be considered: the discrete spectrum containing one mode. This gives the energy an imaginary contribution, namely we have discovered an unstable mode [4,7]. Let us briefly recall, how this appears. The eigenvalue equation

$$\left(\triangle_2\right)_i^a h_{aj} = \alpha h_{ij} \tag{29}$$

can be studied with the Regge-Wheeler method. The perturbations can be divided in odd and even components. The appearance of the unstable mode is governed by the gravitational field component h_{11}^{even}. Explicitly

$$-E^2 H\left(r\right)$$

$$= -\left(1-\frac{2MG}{r}\right)\frac{d^2H\left(r\right)}{dr^2} + \left(\frac{2r-3MG}{r^2}\right)\frac{dH\left(r\right)}{dr} - \frac{4MG}{r^3}H\left(r\right), \tag{30}$$

where

$$h_{11}^{even}\left(r,\vartheta,\phi\right) = \left[H\left(r\right)\left(1-\frac{2m}{r}\right)^{-1}\right]Y_{00}\left(\vartheta,\phi\right) \tag{31}$$

and $E^2 > 0$. Eq.(30) can be transformed into

$$\mu = \frac{\int\limits_0^{\bar{y}} dy\left[\left(\frac{dh(y)}{dy}\right)^2 - \frac{3}{2\rho(y)^3}h^2\left(y\right)\right]}{\int\limits_0^{\bar{y}} dy h^2\left(y\right)}, \tag{32}$$

where μ is the eigenvalue, y is the proper distance from the throat in dimensionless form. If we choose $h\left(\lambda, y\right) = \exp\left(-\lambda y\right)$ as a trial function we numerically obtain $\mu = -.701626$. In terms of the energy square we have

$$E^2 = -.175\,41/\left(MG\right)^2 \tag{33}$$

to be compared with the value $E^2 = -.19/(MG)^2$ of Ref. [4]. Nevertheless, when we compute the eigenvalue as a function of the distance y, we discover that in the limit $\bar{y} \to 0$,

$$\mu \equiv \mu(\lambda) = \lambda^2 - \frac{3}{2} + \frac{9}{8}\left[\bar{y}^2 + \frac{\bar{y}}{2\lambda}\right]. \tag{34}$$

Its minimum is at $\tilde{\lambda} = \left(\frac{9}{32}\bar{y}\right)^{\frac{1}{3}}$ and

$$\mu\left(\tilde{\lambda}\right) = 1.287\,8\bar{y}^{\frac{2}{3}} + \frac{9}{8}\bar{y}^2 - \frac{3}{2}. \tag{35}$$

It is evident that there exists a critical radius where μ turns from negative to positive. This critical value is located at $\rho_c = 1.113\,4$ to be compared with the value $\rho_c = 1.445$ obtained by B. Allen in [23]. What is the relation with the large number of wormholes? As mentioned in Ref. [17], when the number of wormholes grows, the space available for every single wormhole has to be reduced to avoid overlapping of the wave functions. Consider the simple case of two wormholes covering the hypersurface Σ, namely $\Sigma = \Sigma_1 \cup \Sigma_2$, $\Sigma_1 \cap \Sigma_2 = \emptyset$. The second property assures that the boundaries and the support of the wave functional do not overlap, in agreement with the WKB approximation. $\Sigma_1$and Σ_2 have topology $S^2 \times R^1$ with boundaries $\partial\Sigma_1^{\pm}$ and $\partial\Sigma_2^{\pm}$ with respect to each bifurcation surface. The total Hamiltonian, in this case, is $H_T = H_1 + H_2$, i.e. (here we are looking at boundary terms, because in this discussion they are the only relevant ones)

$$H_T = 2H = \frac{1}{8\pi l_p^2}\left[2\int_{S_+} d^2x\sqrt{\sigma}\left(k - k^0\right) - 2\int_{S_-} d^2x\sqrt{\sigma}\left(k - k^0\right)\right]$$

$$= \frac{1}{8\pi l_p^2}\left[2\left(r\left[1 - \left|r_{,y}\right|\right]\right)_{y=y_+} - 2\left(r\left[1 - \left|r_{,y}\right|\right]\right)_{y=y_-}\right]$$

$$= \frac{1}{8\pi l_p^2}\left[2r_+\left(1 - \sqrt{1 - \frac{2Ml_p^2}{r_+}}\right) - 2r_-\left(1 - \sqrt{1 - \frac{2Ml_p^2}{r_-}}\right)\right]$$

$$= \frac{1}{8\pi l_p^2}\left[2r_+\left(1 - \sqrt{1 - \frac{2Ml_{2w}^2}{2r_+}}\right) - 2r_-\left(1 - \sqrt{1 - \frac{2Ml_{2w}^2}{2r_-}}\right)\right]$$

$$= \frac{1}{8\pi l_p^2}\left[R_+\left(1-\sqrt{1-\frac{2Ml_{2_w}^2}{R_+}}\right)-R_-\left(1-\sqrt{1-\frac{2Ml_{2_w}^2}{R_-}}\right)\right]. \tag{36}$$

This means that the total quasilocal energy is the same of a single wormhole with boundaries satisfying the relation $R_\pm = 2r_\pm$, or in other words the value of $R_\pm$ in presence of two wormholes is divided by two. Note that $R_\pm$ are the boundary values corresponding to the single wormhole case. This implies that if we put more and more wormholes, say N_w, the initial boundary located at $R_\pm$ will be reduced and $G \to N_w G$. This boundary reduction is important, because it is related to the disappearing of the unstable mode. Let us see how. If we fix the initial boundary at $R_\pm$, then in presence of N_w wormholes, it will be reduced to $R_\pm/N_w$. This means that boundary conditions are not fixed at infinity, but at a certain finite radius and the ADM mass term is substituted by the quasilocal energy expression under the condition of having symmetry with respect to each bifurcation surface. The effect on the unstable mode is clear: as N_w grows, the boundary radius reduces more and more until it will reach the critical value ρ_c below which no negative mode will appear corresponding to a critical wormholes number N_{w_c}. To this purpose, suppose to consider N_w wormholes and assume that there exists a covering of Σ such that $\Sigma = \bigcup_{i=1}^{N_w} \Sigma_i$, with $\Sigma_i \cap \Sigma_j = \emptyset$ when $i \neq j$. Each Σ_i has the topology $S^2 \times R^1$ with boundaries $\partial\Sigma_i^\pm$ with respect to each bifurcation surface. On each surface Σ_i, quasilocal energy gives [19–22]

$$E_{i\ \mathrm{ql}} = \frac{1}{8\pi l_p^2}\int_{S_{i+}} d^2x\sqrt{\sigma}\left(k-k^0\right) - \frac{1}{8\pi l_p^2}\int_{S_{i-}} d^2x\sqrt{\sigma}\left(k-k^0\right). \tag{37}$$

Thus if we apply the same procedure of the single case on each wormhole, we obtain

$$E_{i\ \mathrm{ql}} = \left(E_{i+}-E_{i-}\right) = l_p^{-2}\left(r\left[1-\left|r_{,y}\right|\right]\right)_{y=y_{i+}} - l_p^{-2}\left(r\left[1-\left|r_{,y}\right|\right]\right)_{y=y_{i-}}. \tag{38}$$

Note that the total quasilocal energy is zero for boundary conditions symmetric with respect to *each* bifurcation surface $S_{0,i}$. If we assume this kind of symmetry for boundary conditions, it is immediate to recognize that the vanishing of the boundary term is guaranteed beyond the semiclassical approximation, because every term on one wedge of the hypersurface Σ_i will be compensated by the term on the other wedge of the same hypersurface Σ_i, giving

therefore zero energy contribution. We are interested in a large number of wormholes, each of them contributing with a term of the type $E_{i\ \mathrm{ql}}$. If the wormholes number is N_w, we obtain (semiclassically, i.e., without self-interactions)

$$H_{tot}^{N_w} = H^1 + H^2 + \ldots + H^{N_w}. \tag{39}$$

Thus the total energy for the collection is

$$E_{|2}^{tot} = N_w H_{|2}.$$

The same happens for the trial wave functional which is the product of N_w t.w.f.. Thus

$$\Psi_{tot}^{\perp} = \Psi_1^{\perp} \otimes \Psi_2^{\perp} \otimes \ldots\ldots \Psi_{N_w}^{\perp} = \mathcal{N} \exp N_w \left\{ -\frac{1}{4l_p^2} \left[\left\langle (g - \bar{g}) K^{-1} (g - \bar{g}) \right\rangle_{x,y}^{\perp} \right] \right\}. \tag{40}$$

Thus for the N_w wormholes, one gets

$$\Delta E_{N_w}(x, \Lambda) \sim N_w \frac{V}{32\pi^2} \Lambda^4 x^2 \ln x, \tag{41}$$

where we have defined the usual scale variable $x = 3M/(l_p \Lambda^2)$. Then at one loop the cooperative effects of wormholes behave as one *macroscopic single* field multiplied by N_w^2, but without the unstable mode. At the minimum, $\bar{x} = e^{-\frac{1}{2}}$

$$\Delta E(\bar{x}) = -N_w \frac{V}{64\pi^2} \frac{\Lambda^4}{e}. \tag{42}$$

This means that we have obtained a minimum of the effective energy away by the flat space, indicating that another configuration has to be considered for the ground state of quantum gravity. Let us examine the implications on the area quantization, entropy and the cosmological constant.

III. AREA SPECTRUM AND ENTROPY

Bekenstein made the proposal that a black hole *does* have an entropy proportional to the area of its horizon [24]

$$S_{bh} = const \times A_{hor}. \tag{43}$$

In particular, in natural units one finds that the proportionality constant is set to $1/4G = 1/4l_p^2$, so that the entropy becomes

$$S = \frac{A}{4G} = \frac{A}{4l_p^2}. \tag{44}$$

Following Bekenstein's proposal on the quantization of the area for nonextremal black holes we have

$$A_n = \alpha l_p^2 (n + \eta) \qquad \eta > -1 \qquad n = 1, 2, \ldots \tag{45}$$

Many attempts to recover the area spectrum have been done, see Refs. [26,27] for a review. Note that the appearance of a discrete spectrum is not so trivial. Indeed there are other theories, based on spherically symmetric metrics in a mini-superspace approach, whose mass spectrum is continuous [28,29]. The area is measured by the quantity

$$A(S_0) = \int_{S_0} d^2x \sqrt{\sigma}. \tag{46}$$

σ is the two-dimensional determinant coming from the induced metric σ_{ab} on the boundary S_0. We would like to evaluate the mean value of the area

$$A(S_0) = \frac{\left\langle \Psi_F \left| \hat{A} \right| \Psi_F \right\rangle}{\langle \Psi_F | \Psi_F \rangle} = \frac{\left\langle \Psi_F \left| \widehat{\int_{S_0} d^2x \sqrt{\sigma}} \right| \Psi_F \right\rangle}{\langle \Psi_F | \Psi_F \rangle}, \tag{47}$$

computed on

$$|\Psi_F\rangle = \Psi_1^{\perp} \otimes \Psi_2^{\perp} \otimes \ldots\ldots \Psi_{N_w}^{\perp}. \tag{48}$$

Since we are working with spherical symmetric wormholes we consider $\sigma_{ab} = \bar{\sigma}_{ab} + \delta\sigma_{ab}$, where $\bar{\sigma}_{ab}$ is such that $\int_{S_0} d^2x \sqrt{\bar{\sigma}} = 4\pi \bar{r}^2$ and $\bar{r}$ is the radius of S_0. To the lowest level in the expansion of σ_{ab} we obtain that

$$A(S_0) = \frac{\left\langle \Psi_F \left| \hat{A} \right| \Psi_F \right\rangle}{\langle \Psi_F | \Psi_F \rangle} = 4\pi \bar{r}^2. \tag{49}$$

Suppose to consider the mean value of the area A computed on a given *macroscopic* fixed radius R. On the basis of our foam model, we obtain $A = \bigcup_{i=1}^{N} A_i$, with $A_i \cap A_j = \emptyset$ when $i \neq j$. Thus

$$A = 4\pi R^2 = \sum_{i=1}^{N} A_i = \sum_{i=1}^{N} 4\pi \bar{r}_i^2. \tag{50}$$

In Refs. [18] we have considered, as a first approximation, the limit $\bar{r}_i \to l_p$ and we have obtained

$$A = NA_{l_p} = N4\pi l_p^2. \tag{51}$$

Nevertheless an improvement of Eq.(51) is possible if we introduce a scale variable $x_i = \bar{r}_i / l_p$ which leads to

$$A = 4\pi l_p^2 \sum_{i=1}^{N} x_i^2 = 4\pi l_p^2 N \overline{x^2} = 4\pi l_p^2 N\alpha. \tag{52}$$

Thus the number α appearing in Eq.(45), here comes from an averaging process. Note that the 4π factor is a consequence of the S^2 wormhole topology which is an intrinsic feature of our foam model. Comparison of Eq.(52)with the Bekenstein area spectrum [26] gives

$$4\pi l_p^2 N\alpha = 4l_p^2 N \ln 2. \tag{53}$$

This fixes the coefficient α to

$$\frac{\ln 2}{\pi} = \alpha \tag{54}$$

and the entropy is

$$S = \frac{A}{4l_p^2} = \frac{4l_p^2 N \ln 2}{4l_p^2} = N \ln 2. \tag{55}$$

N is such that $N \geq N_{w_c}$ and N_{w_c} is the critical wormholes number above which we have the stability of our foam model. On the other hand if we apply the same reasoning of Refs. [30,31], applied to the quantity

$$\frac{A}{4\pi l_p^2} = \sum_{i=1}^{N} x_i^2, \tag{56}$$

produces an extra-factor of the form $\ln 2/\pi$, when compared with the Hawking's coefficient $1/4$. This factor can be absorbed by choosing a suitable normalization constant when we apply the partition of the integer N. In any case we are led to the Bekenstein-Hawking relation between entropy and area [24,25]

$$S = \frac{A}{4l_p^2}. \tag{57}$$

We can use Eq.(52) to compute the entropy for some specific geometries, for example, the Schwarzschild geometry

$$S = \frac{4\pi\left(2MG\right)^2}{4G} = 4\pi M^2 G = 4\pi M^2 l_p^2 = N \ln 2. \tag{58}$$

Thus the Schwarzschild black hole mass is *quantized* in terms of l_p giving therefore the relation

$$M = \frac{\sqrt{N}}{2l_p}\sqrt{\frac{\ln 2}{\pi}}, \tag{59}$$

which is in agreement with the results presented in Refs. [32–38]. This implies also that the level spacing of the transition frequencies is

$$\omega_0 = \Delta M = \left(8\pi M l_p^2\right)^{-1} \ln 2 \tag{60}$$

and the Schwarzschild radius is *quantized* in terms of l_p. Indeed

$$R_S = 2MG = 2Ml_p^2 = \sqrt{N} l_p \sqrt{\frac{\ln 2}{\pi}}. \tag{61}$$

IV. THE COSMOLOGICAL CONSTANT

Einstein introduced his cosmological constant Λ_c in an attempt to generalize his original field equations. The modified field equations are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda_c g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{62}$$

By redefining

$$T^{\mu\nu}_{tot} \equiv T^{\mu\nu} - \frac{\Lambda_c}{8\pi G} g^{\mu\nu}, \tag{63}$$

one can regain the original form of the field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}, \tag{64}$$

at the prize of introducing a vacuum energy density and vacuum stress-energy tensor

$$\rho_\Lambda = \frac{\Lambda_c}{8\pi G}; \qquad T^{\mu\nu}_\Lambda = -\rho_\Lambda g^{\mu\nu}. \tag{65}$$

If we look at the Hamiltonian in presence of a cosmological term, we have the expression

$$H = \int_\Sigma d^3x (N \left(\mathcal{H} + \rho_\Lambda \sqrt{g} \right) + N^i \mathcal{H}_i), \tag{66}$$

where $\mathcal{H}$ is the usual Hamiltonian density defined without a cosmological term. We know that the effect of vacuum fluctuation is to inducing a cosmological term. Indeed by looking at Eq.(42), we have that

$$\frac{\langle \Delta H \rangle}{V} = -N_w \frac{\Lambda^4}{64 e \pi^2}. \tag{67}$$

On the other hand, the WDW equation in presence of a cosmological constant is

$$\left[\frac{16\pi l_p^2}{\sqrt{g}} G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{16\pi l_p^2} \left(R - 2\Lambda_c \right) \right] \Psi \left[g_{ij} \right] = 0. \tag{68}$$

By integrating over the hypersurface Σ and looking at the expectation values computed on the state $|\Psi_F\rangle$ of Eq.(48) the WDW equation becomes

$$\left\langle \Psi_F \left| \int_\Sigma d^3x \left[\frac{16\pi l_p^2}{\sqrt{g}} G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{16\pi l_p^2} R \right] \right| \Psi_F \right\rangle$$

$$= \left\langle \Psi_F \left| - \frac{\Lambda_c}{8\pi l_p^2} \int_\Sigma d^3x \sqrt{g} \right| \Psi_F \right\rangle = -\frac{\Lambda_c}{8\pi l_p^2} \int_\Sigma d^3x \sqrt{g} = -\frac{\Lambda_c}{8\pi l_p^2} V_c. \tag{69}$$

V_c is the cosmological volume. The first term of Eq.(69) is formally the same that generates the vacuum fluctuation (67). Thus, by comparing the second term of Eq.(69) with Eq.(67), we have

$$-\frac{\Lambda_c}{8\pi l_p^2}V_c = -N_w\frac{\Lambda^4}{64e\pi^2}V_w. \qquad (70)$$

Therefore

$$\Lambda_c = N_w^2\frac{\Lambda^4 l_p^2}{V_c 8e\pi}V_w. \qquad (71)$$

The cosmological volume has to be rescaled in terms of the wormhole radius, in such a way to obtain that $V_c \to N_w^3 V_w$ and we have rescaled the Planck length as in page 12. This is the direct consequence of the boundary rescaling, namely $R_\pm \to R_\pm/N_w$. Thus

$$\Lambda_c = \frac{\Lambda^4 l_p^2}{N_w 8e\pi}. \qquad (72)$$

This is the value of the *induced cosmological constant*. On the other hand, if we apply the area quantization procedure of Eq.(58) to the de Sitter geometry, one gets

$$S = \frac{3\pi}{l_p^2\Lambda_c} = N\ln 2, \qquad (73)$$

that is[2]

$$\frac{3\pi}{\ln 2 l_p^2 N} = \Lambda_c. \qquad (74)$$

Thus the cosmological constant Λ is "*quantized*" in terms of l_p. Note that when the wormholes number N is quite "*large*", $\Lambda \to 0$. We could try to see what is the rate of change between an early universe value of the cosmological constant and the value that we observe. In inflationary models of the early universe is assumed to have undergone an early phase with a large effective $\Lambda \sim (10^{10} - 10^{11}GeV)^2$ for GUT era inflation, or $\Lambda \sim (10^{16} - 10^{18}GeV)^2$ for

[2]A relation relating Λ and G, via an integer N appeared also in Ref. [39]. Nevertheless in Ref. [39], N represents the number of scalar fields and the bound from above and below

$$|2GN\Lambda/3 - 2| \geq \sqrt{3}$$

comes into play, instead of the equality (74).

Planck era inflation. A subsequent phase transition would then produce a region of space-time with $\Lambda \leq (10^{-42}GeV)^2$, i.e. the space in which we now live. For GUT era inflation, we have (we are looking only at the order of magnitude)

$$10^{20} - 10^{22}GeV^2 = \frac{1}{N}10^{38}GeV^2 \to N = 10^{16} - 10^{18}, \tag{75}$$

while for Planck era inflation we have

$$10^{32} - 10^{36}GeV^2 = \frac{1}{N}10^{38}GeV^2 \to N = 10^{6} - 10^{2}, \tag{76}$$

to be compared with the value of $(10^{-42}GeV)^2$ which gives a wormholes number of the order of

$$10^{-84}GeV^2 = \frac{1}{N}10^{38}GeV^2 \to N = 10^{122}. \tag{77}$$

In our model this very huge number represents the maximum wormholes number of Planck size that can be stored into an area of radius equal to the cosmological radius. This is in agreement with observational data of Eq.(5). If we compare the previous value of Λ_c with the value of Eq.(74), one gets

$$\Lambda_c = \frac{\Lambda^4 l_p^2}{N_w 8e\pi} = \frac{3\pi}{\ln 2 l_p^2 N_w}, \tag{78}$$

namely we have a constraint on the U.V. cut-off

$$\Lambda^4 = \frac{24e\pi^2}{\ln 2 l_p^4}. \tag{79}$$

The probability to realize a foamy spacetime is measured by

$$\Gamma_{\text{N}-\text{holes}} = \frac{P_{\text{N}-\text{holes}}}{P_{\text{flat}}} \simeq \frac{P_{\text{foam}}}{P_{\text{flat}}}. \tag{80}$$

In a Euclidean time this is

$$P \sim \left| e^{-(\Delta E)(\Delta t)} \right|^2 \sim \left| \exp\left(N_w \frac{\Lambda^4}{e64\pi^2} \right) (V\Delta t) \right|^2. \tag{81}$$

From Eq.(70), we obtain

$$P \sim \left| \exp \left(\frac{\Lambda_c}{8\pi l_p^2} V_c \right) (\Delta t) \right|^2 . \tag{82}$$

To be concrete we can consider again the de Sitter case. Thus

$$\Delta t = 2\pi \sqrt{\frac{3}{\Lambda_c}} \tag{83}$$

and the cosmological volume is given by

$$V_c = \frac{4\pi}{3} \left(\sqrt{\frac{3}{\Lambda_c}} \right)^3 , \tag{84}$$

namely

$$\exp \left(3\pi / l_p^2 \Lambda_c \right) . \tag{85}$$

Thus we recover the Hawking result about the cosmological constant approaching zero [25]. Note that the vanishing of Λ_c is related to the growing of the wormholes number.

V. CONCLUSIONS

In this paper we have continued the investigation of our spacetime foam model presented in Refs. [17,18], where we have obtained a "*quantization*" process in the sense that we can fill spacetime with a given integer number of disjoint non-interacting wormholes. At first look, it seems that our foam model looks promising, since in this framework we have reproduced certain features that a quantum theory of gravity must possess. Nevertheless a lot of points must be clarified. First of all the rôle of the Planckian cutoff that here is computed by comparing a tree level quantity (the entropy of the de Sitter space) with a one-loop quantity (the induced cosmological constant or the Casimir energy). Secondly, the effect of quantum fluctuation has to be inserted in the entropy computation. This could cause a modification of Eqs.(59) and (74) and therefore of estimate (77). On the other hand, as a first consequence we have obtained that the area operator has a discrete spectrum, whose quanta are Planck size wormholes. This is in agreement with the quantized area proposed heuristically by Bekenstein and also with the loop quantum gravity predictions of

Refs. [30,31]. Note that in order to have stability, it is the energy configuration that forces spacetime to be filled with N wormholes of the Planckian size. Since the area is related to the entropy via the Bekenstein-Hawking relation, as a direct application, a "*mass quantization*" of a Schwarzschild black hole whose mass is M is obtained, in agreement with Refs. [32–38]. The second consequence of our model is the generation of a positive cosmological constant induced by vacuum fluctuations. Due to the uncertainty relation

$$\Delta E \propto \frac{A}{L^4} \propto -N_w \frac{V}{64\pi^2} \frac{\Lambda^4}{e} \propto A\Lambda^4. \tag{86}$$

The negative fourth power of the cutoff (or the inverse of the fourth power of the region of dimension L) is a clear signal of a Casimir-like energy generated by vacuum fluctuations. As a consequence a *positive cosmological constant* is induced by such fluctuations.

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